1 Facts to Remember

1. Riesz Theorem: There exists a 1-1 correspondence between linear functionals $F \in V'$ and vectors $f \in V$ such that $F(\phi) = (f, \phi)$. (Recall that (f, ϕ) is the inner product of f and ϕ defined in §1.2-1.3 of the text.)
2. *Linear operator.* $A(c_1\psi_1 + c_2\psi_2) = c_1(A\psi_1) + c_2(A\psi_2)$
3. $M|\psi >= |\phi > \quad \Rightarrow \quad \sum_i M_{ij} a_j = b_i$
4. *Self-Adjoint* $\Rightarrow A = A^\dagger$ (adjoint is the complex transpose)
5. An operator A is Hermitian $\Rightarrow$ eigenvalues are real ($\in \mathbb{R}$).
6. Eigenvectors corresponding to distinct eigenvalues of a Hermitian operator are orthogonal.
7. Orthonormal $\Rightarrow (\phi_i, \phi_j) = \delta_{ij}$.
8. $\{\phi_i\}$ is complete $\quad \Rightarrow \quad$ any vector $|v >= \sum_i v_i|\phi_i > \;\; \Rightarrow \;\; v_i =< \phi_i|v >$
9. 'Vectors' in V are $|\,\phi\rangle$ "ket"; 'Vectors' in V' are $< F|$, "bra" are known as linear functionals.
10. A linear space V consists of all possible linear combinations of vectors in V.

The points above are useful as we look toward constructing Hilbert spaces and build mathematical infrastructure for the foundations of quantum mechanics.

2 What is a Hilbert space?

2.1 Definition

The following steps describe how we shall construct a Hilbert space.

1. Consider a linearly independent basis $\{\phi_n\}$ for $n = 1, 2, ...$ that spans a linear vector space V. Construct V from all possible linear combinations of $\{\phi_n\}$. Any vector ψ in V is of the form
$$\psi = \sum_n c_n \phi_n \tag{1}$$
2. Next, we enlarge V by adding convergent infinite sequences of vectors (i.e. limit points). We do this in an 'analytical' approach by analysis of norms of convergent infinite series. [1]
3. Consider
$$\psi = \sum_n^i c_n \phi_n \tag{2}$$
where the vector corresponding to $i \to \infty \notin V$ but the vector $i \to \infty \in \mathbb{H}$. $\mathbb{H}$ will be called a *Hilbert space.*

2.2 Comments and More...

1. Recall the Riesz Theorem applied to our new construction of a Hilbert space. $\exists$ a $1-1$ correspondence between vectors in $\mathbb{H}$ and its dual $\mathbb{H}'$ composed of linear functionals.
2. *A Practical Construction.* Given Ξ a linear space with basis $\{\phi_n\}$. An arbitrary element in Ξ is denoted
$$\xi = \sum_n c_n \phi_n \quad \text{(a column vector)}^2 \tag{3}$$

[1] What is meant by "convergent" in a space of vectors? Examine the norm for a sequence ψ as follows. $\{\psi_i\} \longrightarrow_{i\to\infty} \chi$ if and only if $\lim_{i\to\infty} \quad \|\psi_i - \chi\| = 0$. By carrying this out we can enlarge V.

[2] norm and inner product are undefined for many elements

Definition 1 *A Hilbert space* $\mathbb{H} \subset \Xi$ *is defined by the constraint that* $h = \sum_n c_n \phi_n \in \mathbb{H}$ *if and only if* $(h, h) = \sum_n |c_n|^2 < \infty$*, i.e. the inner product of* h *with itself is finite.*

Definition 2 $\mathbb{H}^\times$*, the conjugate space, is all vectors* $f = \sum_n b_n \phi_n$ *for which* $(f, h) = \sum_n b_n^* c_n$ *is convergent for every* $h \in \mathbb{H}$*, and* (f, h) *is a continuous linear functional on* $\mathbb{H}$*.*

Consider the choice of h such that 'phase of c_n' $\equiv$ 'phase of b_n' $\Rightarrow b_n^* c_n > 0$ and real. Thus

$\Rightarrow (f, h) = \sum_n b_n^* c_n$ converges if $|b_n| \to 0$ as rapidly as $|c_n| \to 0$ as $n \to \infty$ because $\sum_n |c_n|^2 < \infty$.

$\Rightarrow \sum_n |b_n|^2$ is convergent $\Rightarrow f \in \mathbb{H}$

$\therefore$ a Hilbert space $\mathbb{H} \equiv \mathbb{H}'$ (Hilbert space is equivalent to its conjugate)

2.3 Rigged Hilbert Space

Define Ω as a space consisting of all vectors of the form $\omega = \sum_n u_n \phi_n$ such that $\sum_n |u_n|^2 n^m < \infty$ (finite). Consider $\Omega \subset \mathbb{H}$ called a "nuclear space". Then,

Definition 3 $\Omega^\times$ *the conjugate of* Ω *consists of* $\sigma = \sum_n v_n \phi_n$ *such that* $(\sigma, \omega) = \sum_n v_n^* u_n$ *is convergent for all* $\omega \in \Omega$ *and* $(\sigma, \cdot)$ *is a continuous linear functional on* Ω*.*

One can observe that $\Omega^\times$ is much larger than Ω. Thus consider the following $\Omega \subset \mathbb{H} \subset \Omega^\times$. This is called a *rigged Hilbert space.*

GOAL: Enlarge $\mathbb{H}$ to a larger space. That larger space is the rigged Hilbert space.

There are many examples of rigged-Hilbert-space triplets. Let's look at an example (taken from the text §1.4, p. 28). If Ξ is taken to be a linear vector space of functions of one variable, then a Hilbert space is formed by those functions that are square-integrable. $\mathbb{H}$ is made up of those functions $\psi(x)$ where

$$(\psi, \psi) = \int_{-\infty}^{+\infty} |\psi(x)|^2 dx < \infty \text{ (finite)} \tag{4}$$

The extended space $\Omega^\times$, conjugate to Ω consists of χ where

$$(\chi, \phi) = \int_{-\infty}^{+\infty} \chi(x)^* \phi(x) dx < \infty \text{ (finite for all } \phi \in \Omega) \tag{5}$$

The equation in (5) is much better for constructing quantum mechanics. This will be discussed in upcoming lectures.

END OF LECTURE 1