Mathematical Methods for Physicists I
Fall 2003

Name:

HUID:

SOLUTIONS

Quiz 10 <u>Directions</u>: This quiz is open notes/book. Please show all work.

Show that every complex second order polynomial with real coefficients is infinitely differentiable.

The general form for any complex second order polynomial is given as $f(z) = az^2 + bz + c$, where $a, b, c \in \mathbb{R}$. The goal is to show that f(z) is infinitely differentiable, i.e. that f(z) has an infinite number of derivatives that exist. Recall that in the theory of complex analysis in one variable, the derivative of a complex valued function exists if the function $f(z)$ satisfies the Cauchy-Riemann equations.

Given $f(z) = u(x, y) + iv(x, y)$ where $u(x, y), v(x, y)$ are real-valued functions. Then $f'(z)$ exists if

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (Cauchy-Riemann Equations).

Recall that $z = x + iy$. Using this fact let's rewrite $f(z) = az^2 + bz + c$ as follows:

$f(z) = az^2 + bz + c = a(x + iy)^2 + b(x + iy) + c = (a(x^2 - y^2) + bx + c) + i(2axy + by)$.

We can identify $Ref(z) = u(x, y) = a(x^2 - y^2) + bx + c$ and $Imf(z) = v(x, y) = 2axy + by$. Now take the necessary partial derivatives to verify the Cauchy-Riemann equations.

Thus,
$\frac{\partial u}{\partial x} = 2ax + b = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = -2ay = -\frac{\partial v}{\partial x}$

Therefore, $f'(z)$ exists and $f'(z) = 2az + b$. Remember the goal of this problem was to show that $f(z)$ is infinitely differentiable. This requires showing that every derivative satisfies the Cauchy-Riemann equations. Continuing in the manner above for $f'(z)$ yields

$f'(z) = 2az + b = 2a(x + iy) + b = (2ax + b) + i2ay$. Let $u = 2ax + b$ and $v = 2ay$.

Thus,
$\frac{\partial u}{\partial x} = 2a = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$

Hence $f''(z)$ exists and $f''(z) = 2a$. To show that $f'''(z)$ exists we must satisfy the Cauchy-Riemann equations again. So,

Then for $f''(z) = 2a$, let $u = 2a$ and $v = 0$.

$\frac{\partial u}{\partial x} = 0 = \frac{\partial v}{\partial y}$

$\frac{\partial u}{\partial y} = 0 = -\frac{\partial v}{\partial x}$

Hence $f'''(z)$ exists and $f'''(z) = 0$. Since every derivative greater than degree three is 0, $u = 0$ and $v = 0$ which satisfies the Cauchy-Riemann equations for all z.

$\therefore f(z)$ is infinitely differentiable. $\star\star$