

# Addition of Angular Momenta, Clebsch-Gordan Coefficients and the Wigner-Eckart Theorem

(à la Speedy Gonzales)

The following is a lightning introduction to using symmetries to their fullest. The presentation in Ref. [1] and [2] is followed, but any other source with at least as complete a coverage will be at least just as good—provided you understand the basic mechanism, which this note attempts to clarify.

## 1. Summing Angular Momenta

First of all, a disclaimer: in what follows, we shall use the symbol  $\hat{\mathbf{J}}$  for any triple of operators,  $\hat{J}_1, \hat{J}_2, \hat{J}_3$  which satisfy the “standard angular momentum commutation relations” (a.k.a. “standard angular momentum algebra” or the “ $su(2)$  algebra”):

$$[\hat{J}_j, \hat{J}_k] = i\epsilon_{jkl}\hat{J}_l, \quad j, k, l = 1, 2, 3 \text{ or } x, y, z, \quad (1.1)$$

and all results will be equally general in scope. (Summation over repeated indices, such as  $l$  in (1.1), is implied!)

### 1.1. Vector-Addition of Angular Momenta

In *any* quantum mechanics application, these operators act on wave-functions, that we may label by the eigenvalues of  $\hat{\mathbf{J}}^2 = \sum_i \hat{J}_i^2$  and *one* of the components, say  $\hat{J}_3$ . The axis in which  $\hat{J}_3$  points is called the “quantization axis”, and its choice is *arbitrary*. This ‘axis’ makes sense as a direction in the usual space only if  $\hat{\mathbf{J}}$  is an actual angular momentum vector operator—orbital, spin or a combination. If  $\hat{\mathbf{J}}$  was meant to have a different interpretation (see §9.7 and 20 of [3]), then an associated 3-dimensional *abstract* space will be convenient to imagine, in which the  $\hat{\mathbf{J}}$  generate abstract ‘rotations’—but this will have absolutely nothing to do with rotations in the real-life space. These wave-functions we write as  $|j, m\rangle$  and know that

$$\hat{\mathbf{J}}^2 |j, m\rangle = j(j+1) |j, m\rangle, \quad \text{and} \quad \hat{J}_3 |j, m\rangle = m |j, m\rangle. \quad (1.2)$$

It will also be useful to recall that

$$\hat{J}_\pm \stackrel{\text{def}}{=} \hat{J}_1 \pm i\hat{J}_2, \quad (1.3)$$

and that

$$\hat{J}_\pm |j, m\rangle = \sqrt{(j \mp m)(j \pm m + 1)} |j, m \pm 1\rangle \quad (1.4)$$

Until you are completely comfortable with these, read over §5.7–5.9 of [3], or any other source that covers this and work on the examples and problems.

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Given two parts of a system, each with its own ‘angular momentum’,  $\hat{\mathbf{J}}_1$  and  $\hat{\mathbf{J}}_2$ , it makes perfect sense *add* these to vectorial quantities as one usually does with vectors:

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2 . \quad (1.5)$$

To avoid confusion between the numbering of the subsystems (1, 2) and the numbering of the components of each  $\hat{\mathbf{J}}_i$ , the components will hereafter be labeled by  $x, y, z$  in place of 1, 2, 3—even if the three components of  $\hat{\mathbf{J}}_i$  have nothing to do with the  $x, y, z$  coordinates of the real-life 3-dimensional space. Since (1.5) is a vectorial equation, it must also hold for any one of its components, and so in particular also

$$\hat{J}_z = \hat{J}_{1z} + \hat{J}_{2z} . \quad (1.6)$$

### 1.2. Multiplication of Wave-Functions

Given  $|j_1, m_1\rangle$  for the first and  $|j_2, m_2\rangle$  for the second part of the system, the wave-function of the two parts together may be written as  $|j_1, m_1\rangle |j_2, m_2\rangle$ , and we have that

$$\hat{J}_z |j_1, m_1\rangle |j_2, m_2\rangle = (\hat{J}_{1z} + \hat{J}_{2z}) |j_1, m_1\rangle |j_2, m_2\rangle = (m_1 + m_2) |j_1, m_1\rangle |j_2, m_2\rangle . \quad (1.7)$$

That is, the projection of the total ‘angular momentum’ equals the sum of the projections of the ‘angular momenta’ of the two parts. Given  $j_i$ , there are  $2j_i + 1$  possible projections of  $m_i$ , and so  $(2j_1 + 1)(2j_2 + 1)$  possible sums,  $m = m_1 + m_2$ , many of which occurring with a multiplicity. For example, when  $j_1 = j_2 = \frac{1}{2}$ ,  $m = 0$  occurs twice: once as  $0 = \frac{1}{2} - \frac{1}{2}$ , and once as  $0 = -\frac{1}{2} + \frac{1}{2}$ . The total number of  $m$ ’s is indeed  $(2 \cdot \frac{1}{2} + 1)(2 \cdot \frac{1}{2} + 1) = 4$ :  $m = -1, 0, 0, +1$ .

Certainly, there will exist a linear combination of the product wave-functions  $|j_1, m_1\rangle |j_2, m_2\rangle$ , which will be eigenstates of  $\hat{\mathbf{J}}^2$  and  $\hat{J}_z$ . It is easy to prove that  $\hat{\mathbf{J}}_1$  and  $\hat{\mathbf{J}}_2$  commute with  $\hat{\mathbf{J}}^2$  and  $\hat{J}_z$ , so that there will in fact exist a basis of simultaneous eigenfunctions of all of four  $\hat{\mathbf{J}}^2, \hat{J}_z, \hat{\mathbf{J}}_1, \hat{\mathbf{J}}_2$ . It is also easy to prove that  $\hat{J}_{1z}$  and  $\hat{J}_{2z}$  do not commute with the square of the total angular momentum  $\hat{\mathbf{J}}^2$ , as it contains terms like  $\hat{J}_{1x}\hat{J}_{2x}$  for example, for which

$$[\hat{J}_{1x}\hat{J}_{2x}, \hat{J}_{1z}] = [\hat{J}_{1x}, \hat{J}_{1z}]\hat{J}_{2x} = (-i\hat{J}_{1y})\hat{J}_{2x} , \quad (1.8)$$

and such terms do not cancel out in the whole expansion of  $[\hat{\mathbf{J}}^2, \hat{J}_{iz}]$ ,  $i = 1, 2$ . This new basis is typically written as  $|j_1, j_2, j; m\rangle$  or often just  $|j, m\rangle$  since  $j_1, j_2$  are known and fixed, while the previous product basis is typically written as  $|j_1, m_1\rangle |j_2, m_2\rangle = |j_1, j_2; m_1, m_2\rangle$  (note the different position of the semi-colons!).

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Now we come to the general question: given a state  $|j_1, j_2; m_1, m_2\rangle$ , how do we express it in terms of the  $|j_1, j_2, j; m\rangle$ ?

The general answer is: easily! And the reason is very simple. The collection of wave-functions  $|j_1, j_2; m_1, m_2\rangle$  is by definition complete, and so an expansion

$$|j_1, j_2, j; m\rangle = \sum_{m_1, m_2} c_{j_1, j_2; m_1, m_2}^{j, m} |j_1, j_2; m_1, m_2\rangle \quad (1.9)$$

must always be possible; the coefficients  $c_{j_1, j_2; m_1, m_2}^{j, m}$  are called the Clebsch-Gordan coefficients (a.k.a. Wigner coefficients, or—up to a slight redefinition—the 3- $j$  symbols).

Obviously, we must allow  $j, m$  to take on all allowed values, and in doing so, we obtain another complete set of wave-function: the product-basis wave-functions are simultaneous eigenvalues of  $\hat{\mathbf{J}}_1^2, \hat{\mathbf{J}}_2^2, \hat{J}_{1z}, \hat{J}_{2z}$ , while the new basis wave-functions are simultaneous eigenvalues of  $\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_1^2, \hat{\mathbf{J}}_2^2, \hat{J}_z$ . Clearly, this new complete basis can be made (and always will be assumed to have been made) orthonormal via the Gram-Schmidt orthonormalization algorithm<sup>1)</sup>.

### 1.3. Clebsch-Gordan Coefficients

To determine the Clebsch-Gordan coefficients, we can use the orthonormality of the  $|j_i, m_i\rangle$ . Simply multiply (1.9) by  $\langle j_1, j_2; m'_1, m'_2 | = \langle j_1, m'_1 | \langle j_2, m'_2 |$

$$\langle j_1, j_2; m'_1, m'_2 | j_1, j_2, j; m \rangle = c_{j_1, j_2; m'_1, m'_2}^{j, m}, \quad (1.10)$$

where  $j_1, j_2$  are fixed as originally given. For a variety of historical reasons, these coefficients are also written as

$$c_{j_1, j_2; m_1, m_2}^{j, m} = \langle j_1, j_2; m_1, m_2 | j, m \rangle = (j, j_1, j_2 | m, m_1, m_2), \quad (1.11)$$

and also

$$c_{j_1, j_2; m_1, m_2}^{j, m} = (-1)^{j_1 - j_2 + m} \sqrt{2j+1} \begin{pmatrix} j_1 & j_2 & j \\ m_1 & m_2 & -m \end{pmatrix}, \quad (1.12)$$

where the last, matrix-like symbol is called the 3- $j$  symbol.

As both bases,  $\{|j_1, j_2; m_1, m_2\rangle\}$  and  $\{|j_1, j_2, j; m\rangle\}$  are orthonormal by construction, the transformation (1.9) must be unitary, and its inverse is

$$|j_1, j_2; m_1, m_2\rangle = \sum_{j, m} \langle j, m | j_1, j_2; m_1, m_2 \rangle |j_1, j_2, j; m\rangle, \quad (1.13)$$

where

$$\langle j, m | j_1, j_2; m_1, m_2 \rangle = \langle j_1, j_2; m_1, m_2 | j, m \rangle^*. \quad (1.14)$$

As it turns out, the Clebsch-Gordan coefficients can and are always chosen to be real, so that we may drop the conjugation symbol here.

Thus, using Eqs. (1.9) or (1.13), one can freely toggle between the two bases—provided the Clebsch-Gordan coefficients are known.

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<sup>1)</sup> If you forgot how to do that, see for example § 9.3 of Ref. [4].

It is easy to show that  $\langle j_1, j_2; m_1, m_2 | j, m \rangle = 0$  unless  $m = m_1 + m_2$ —this simply reflects (1.6). In addition, the vector equation (1.5) may be represented by a triangular vector-diagram, where the three vectors  $\hat{\mathbf{J}}, \hat{\mathbf{J}}_1, \hat{\mathbf{J}}_2$  form a triangle. It then follows that

$$|\langle \hat{\mathbf{J}}_1^2 \rangle - \langle \hat{\mathbf{J}}_2^2 \rangle| \leq \langle \hat{\mathbf{J}}^2 \rangle \leq |\langle \hat{\mathbf{J}}_1^2 \rangle + \langle \hat{\mathbf{J}}_2^2 \rangle| , \quad (1.15)$$

It is a bit cumbersome to derive from this relation one which is linear in  $j, j_1, j_2$ . Instead, we note that the maximum possible value for  $m$  is  $m_1|_{\max} + m_2|_{\max} = j_1 + j_2$ . Since this must be the (maximal) projection of the maximal possible  $\hat{\mathbf{J}}$ , we conclude that  $j \leq j_1 + j_2$ . On the other hand, the minimal possible value for  $m$  is  $|m_1|_{\max} - m_2|_{\max}| = |j_1 - j_2|$ . Since this must be the (minimal) projection of the minimal possible  $\hat{\mathbf{J}}$ , we conclude that  $|j_1 - j_2| \leq j$ . Thus:

$$|j_1 - j_2| \leq j \leq j_1 + j_2 . \quad (1.16)$$

Therefore, it must be that  $\langle j, m | j_1, j_2; m_1, m_2 \rangle = 0$  unless the “triangle relation” (1.16) is satisfied.

Independently, this can be verified also as follows. For a given  $j$ , there are  $2j+1$  possible projections  $m$ . The total number of possible  $j, m$  pairs must equal the total number of possible  $j_1, m_1, j_2, m_2$  choices. Therefore:

$$\sum_{j=j_{\min}}^{j=j_{\max}} \left( \sum_{m=-j}^j \right) = \left( \sum_{m_1=-j_1}^{j_1} \right) \left( \sum_{m_2=-j_2}^{j_2} \right) , \quad (1.17a)$$

$$\sum_{j=j_{\min}}^{j=j_{\max}} (2j+1) = (2j_1+1)(2j_2+1) , \quad (1.17b)$$

which is satisfied if  $j_{\min} = |j_1 - j_2|$  and  $j_{\max} = j_1 + j_2$ . (Sum this finite arithmetic series using Gauss’s formula  $\sum_{n=n_1}^{n_2} f(n) = [f(n_1) + f(n_2)] \cdot (n_2 - n_1 + 1) / 2$  since the sum of opposite terms (first+last, second+penultimate, ...) always equals  $f(n_1) + f(n_2)$ , and there are  $n_2 - n_1 + 1$  terms, *i.e.*, half that many pairs.)

Now, there exists a general formula [1], rather cumbersome for a human but straightforward for a machine:

$$c_{j_1, j_2; m_1, m_2}^{j, m} = \delta_{m, m_1 + m_2} \cdot \rho_{j_1, j_2}^j \cdot \sigma \cdot \tau , \quad (1.18a)$$

$$\delta_{m, m_1 + m_2} = \begin{cases} 1 & m = m_1 + m_2, \\ 0 & m \neq m_1 + m_2; \end{cases} , \quad (1.18b)$$

$$\rho_{j_1, j_2}^j = \sqrt{\frac{(j_1 + j_2 - j)! (j + j_1 - j_2)! (j_2 + j - j_1)! (2j + 1)}{(j_1 + j_2 + j + 1)!}} , \quad (1.18c)$$

$$\sigma = \sqrt{(j+m)! (j-m)! (j_1+m_1)! (j_1-m_1)! (j_2+m_2)! (j_2-m_2)!} , \quad (1.18d)$$

$$\tau = \sum_r \frac{(-1)^r}{(j_1 - m_1 - r)! (j_2 + m_2 - r)! (j - j_2 + m_1 + r)! (j - j_1 - m_2 + r)! (j_1 + j_2 - j - r)! r!} , \quad (1.18e)$$

where the sum over  $r$  extends over all values for which none of the factors in the parentheses vanish. This makes the sum finite. Alternatively, use that  $0! = 1$  and that  $(-n)! = \Gamma(1 - n) = \infty$  for  $n = 1, 2, \dots$ , so such terms contribute nothing to the sum.

Useful relations include:

**Completeness:**

$$\sum_{m_1, m_2} c_{j_1, j_2; m_1, m_2}^{j, m} c_{j_1, j_2; m_1, m_2}^{j', m'} = \delta_{j, j'} \delta_{m, m'} , \quad (1.19a)$$

$$\sum_{j, m} c_{j_1, j_2; m_1, m_2}^{j, m} c_{j_1, j_2; m'_1, m'_2}^{j, m} = \delta_{m_1, m'_1} \delta_{m_2, m'_2} . \quad (1.19b)$$

**Symmetry:**

$$c_{j_1, j_2; m_1, m_2}^{j, m} = c_{j_2, j_1; -m_2, -m_1}^{j, -m} = (-1)^{j-j_1-j_2} c_{j_2, j_1; m_2, m_1}^{j, m} . \quad (1.20)$$

## 2. The Wigner-Eckart Theorem

Next, we turn to applying symmetry considerations to matrix elements. To that end, however, we need to know something about how the operators transform with respect to rotations (or ‘rotations’).

### 2.1. Rotation of Operators

Let  $\hat{R}_{\vec{\varphi}} = \exp \{ i \vec{\varphi} \cdot \hat{\mathbf{J}} \}$  be the general operator implementing the transformation which is generated by  $\hat{\mathbf{J}}$ —rotation if  $\hat{\mathbf{J}}$  is indeed the angular momentum operator. These translation operators act as

$$\Psi' = \hat{R}_{\vec{\varphi}} \Psi , \quad \text{and} \quad \hat{Q}' = \hat{R}_{\vec{\varphi}} \hat{Q} \hat{R}_{\vec{\varphi}}^\dagger , \quad (2.1)$$

where  $\hat{R}_{\vec{\varphi}}$  is unitary:  $\hat{R}_{\vec{\varphi}}^\dagger = \hat{R}_{\vec{\varphi}}^{-1}$ . For example, if  $\Psi$  is written as a spherical harmonic and  $\hat{\mathbf{J}}$  is the actual angular momentum vector operator generating rotations, we can choose  $\vec{\varphi} = \varphi \hat{e}_z$ , so  $\hat{R}_{\vec{\varphi}} = \hat{R}_z(\varphi)$  implements rotation about the  $z$  axis by an angle  $\alpha$ . Then  $Y_\ell^m(\theta, \phi)' = \hat{R}_z(\varphi) Y_\ell^m(\theta, \phi) = Y_\ell^m(\theta, \phi + \varphi)$ .

It then follows from general group theory that any operator  $\hat{Q}$  can be decomposed into a sum of terms, each of which transforms as a tensor<sup>2)</sup> of rank  $r$  with respect to the group generated by the  $\hat{\mathbf{J}}$ . When this group is  $SU(2)$ , we can write  $\hat{Q}$  as a linear combination of  $\hat{Q}_q^{(\varrho)}$ , where  $\varrho$  is the rank and  $q = -\varrho, -\varrho+1, \dots, +\varrho$ . That is, the  $(2\varrho+1)$ -tuple of operators  $\{\hat{Q}_q^{(\varrho)}, q = -\varrho, \dots, +\varrho\}$  transforms as components of a rank- $\varrho$  tensor, *i.e.*, have total spin/angular momentum =  $\varrho$ . That is,

$$[\hat{J}_\pm, \hat{Q}_q^{(\varrho)}] = \sqrt{\varrho(\varrho+1) - q(q \pm 1)} \hat{Q}_{q \pm 1}^{(\varrho)} , \quad (2.2a)$$

$$[\hat{J}_z, \hat{Q}_q^{(\varrho)}] = q \hat{Q}_q^{(\varrho)} , \quad (2.2b)$$

which generalizes the algebra (1.1). Note that for  $\varrho = 1$ , this recovers (1.1), which is consistent with the fact that  $\hat{\mathbf{J}}$  is a vector (a.k.a. rank-1 tensor) operator.

<sup>2)</sup> Uh-oh... Forgot what this was? See § 3 of Ref. [4].

## 2.2. The Theorem

Let now  $|\alpha; j, m\rangle$  denote orthonormalized states where  $\alpha$  is a collective label (quantum number) for *all* characteristics that are independent of  $\hat{\mathbf{J}}$ . Then, the Wigner-Eckart theorem states that all matrix operators factorize as follows<sup>3)</sup>

$$\langle \alpha'; j', m' | \hat{Q}_q^{(\varrho)} | \alpha; j, m \rangle = \langle j', m' | j, \varrho; m, q \rangle \langle \alpha'; j' | \hat{Q}^{(\varrho)} | \alpha; j \rangle, \quad (2.3)$$

where  $\langle j', m' | j, \varrho; m, q \rangle$  are the same Clebsch-Gordan coefficients discussed in the previous section, and the “reduced matrix element”  $\langle \alpha'; j' | \hat{Q}^{(\varrho)} | \alpha; j \rangle$  is independent of  $m, m', q$ .

This result determines the relative strength of all matrix elements with identical total spins/angular momenta—the dependence on the projections is an exactly and universally calculable factor, the Clebsch-Gordan coefficient. For example, for two processes mediated by the same operator  $\hat{Q}_q^{(\varrho)}$ :

$$\hat{Q}_q^{(\varrho)} : \begin{cases} |\alpha; j, m\rangle \rightarrow |\alpha'; j', m'\rangle, \\ |\alpha; j, m''\rangle \rightarrow |\alpha'; j', m'''\rangle, \end{cases} \quad (2.4)$$

the relative amplitude is

$$\frac{\langle \alpha'; j', m' | \hat{Q}_q^{(\varrho)} | \alpha; j, m \rangle}{\langle \alpha'; j', m''' | \hat{Q}_q^{(\varrho)} | \alpha; j, m'' \rangle} = \frac{\langle j', m' | j, \varrho; m, q \rangle}{\langle j', m''' | j, \varrho; m'', q \rangle}, \quad (2.5)$$

which is calculable without any knowledge of the operator  $\hat{Q}_q^{(\varrho)}$  other than its transformation property under the group generated by the  $\hat{\mathbf{J}}$ —Eqs. (2.2)!

## 3. Applications

On the qualitative level, (extensions of) this group theory machinery can and has been used to predict new states. For example, in nuclear structure physics, the system for all but the simplest nuclei has too many particles to be solved exactly. Also, it has no exact symmetry (as crystals do) so simplify calculations. Yet, there do exist approximate symmetries of the nucleus when viewed as a collective object (droplet) with a small number of nucleons (protons and/or neutrons) providing perturbations. The group theory machinery has been used to predict the qualitative form of the spectrum of states and allows a Hamiltonian to be written down with relatively few parameters. Fitting these parameters to the lowest-lying states allowed then to predict the higher-lying states and is still pretty much *the only* way of modelling certain heavy nuclei [5].

A little more spectacular application of this has been to particle physics, where the symmetries in question are not of space-time sort. Nevertheless, it is the same group theoretical machinery that predicted the existence of the  $\Omega$  particle, and with precisely the properties subsequently found in experiment (see § 20 of Ref. [3]).

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<sup>3)</sup> Slightly different normalizations and sign conventions for the reduced matrix element will produce a slightly different form of this factorization statement.

On the quantitative side, consider the following problem. A nucleus like  ${}^{49}_{21}\text{Sc}$  (a radioactive isotope of Scandium) has 21 protons and 28 neutrons. It may be described as an inert (spinless) droplet formed of the 20 protons and 28 neutrons (these being the numbers of particles in closed nuclear shells) and a single proton orbiting around it. Consider the case when the proton (with spin- $\frac{1}{2}$ ) orbits with total angular momentum  $j = \frac{3}{2}$ ; the total angular momentum being the sum of orbital and spin angular momentum:  $\hat{\mathbf{j}} = \hat{\mathbf{l}} + \hat{\mathbf{s}}$ . Incident on this proton is an X-ray and the proton absorbs it. Considering only allowed transitions, the matrix element for the interaction process will be proportional to  $\langle j', m' | \vec{r} | \frac{3}{2}, m \rangle$ .

1. What are the possible total angular momenta of the proton after absorption?  $j' = \frac{5}{2}, \frac{3}{2}, \frac{1}{2}$ , since the operator  $\vec{r}$  is a vector, so that  $\varrho = 1$ , and  $|j - \varrho| \leq j' \leq (j + \varrho)$ .
2. What are the possible projections of the total angular momentum?  $m' = m \pm 1$  or  $m' = m$ , depending on the angle between the quantization axis and  $\vec{r}$ . (In a typical experiment, the quantization axis will be determined by a constant background magnetic field.)
3. Assuming the proton stayed in the  $j = j' = \frac{3}{2}$  state, what are the relative probabilities for the various projections? Since probabilities depend on the square of the absolute value of the matrix elements and all other coefficients are the same (not depending on projections of angular momenta), they are:

$$|\langle \frac{3}{2}, m+1 | 1, \frac{3}{2}; 1, m \rangle|^2 : |\langle \frac{3}{2}, m | 1, \frac{3}{2}; 0, m \rangle|^2 : |\langle \frac{3}{2}, m-1 | 1, \frac{3}{2}; -1, m \rangle|^2 . \quad (3.1)$$

For  $m = \pm\frac{3}{2}, \pm\frac{1}{2}$ , this gives the twelve matrix elements, up to a single reduced matrix element! For example, if the initial projection was  $m = \frac{1}{2}$ , it is 6 times more probable that the final projection will grow to  $\frac{3}{2}$ , and 8 times more probable that it will flip to  $-\frac{1}{2}$  than stay as  $\frac{1}{2}$ . That's because

$$\begin{aligned} \langle \frac{3}{2}, \frac{3}{2} | 1, \frac{3}{2}; 1, \frac{1}{2} \rangle &= -\sqrt{2/5} , & \langle \frac{3}{2}, \frac{1}{2} | 1, \frac{3}{2}; 0, \frac{1}{2} \rangle &= \sqrt{1/15} , \\ \langle \frac{3}{2}, -\frac{1}{2} | 1, \frac{3}{2}; -1, \frac{1}{2} \rangle &= \sqrt{8/15} . \end{aligned}$$

That is, 2/5 of the time, the projection will grow to  $\frac{3}{2}$ , 1/15 of the time stay  $\frac{1}{2}$ , and 8/15 of the time flip to  $-\frac{1}{2}$ .

## References

- [1] L.D. Landau and M.E. Lifsic: *Quantum Mechanics* (Pergamon Press, Oxford, 1977, 3rd ed.).
- [2] W. Greiner and B. Müller: *Quantum Mechanics: Symmetries* (Springer-Verlag, Berlin, 1994, 2nd ed.).
- [3] D. Park: *Introduction to the Quantum Theory* (McGraw-Hill, New York, 1992, 3rd ed.).
- [4] G. Arfken: *Mathematical Methods for Physicists* (Academic Press, Inc., Boston, 1985, 3rd ed.).
- [5] T. Hübsch, V. Paar and D. Vretenar: *Spin(6)* Boson-Fermion Dynamical Symmetry for Odd-Odd Nuclei. *Phys. Lett.* **151B** (1985)320;  
T. Hübsch and V. Paar: Extension of Boson-Fermion Dynamical Symmetry to Hypernuclei. *Phys. Lett.* **151B** (1985)1.